A STUDY OF THE OPTIMAL SOPHISTICATION OF A MINE

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ABSTRACT

A basic method of determining the nature of the optimal sophistication of a mine, given the statistical distribution of targets and minesweepers within a complex, is developed. The method is based on simple probability theory and logic and an attempt is made to apply mathematical criteria to the problem. Several different target and minesweeper distributions, as well as minehunting effort, are examined, discussed, and graphically displayed.

The writer wishes to express his appreciation for the invaluable encouragement, assistance and guidance given him by Dr. Charles C. Torrance of the U. S. Naval Postgraduate School without which this study would not have been possible.

Appreciation is also expressed to Dr. Donald Guthrie, Jr. of the U. S. Naval Postgraduate School, who acted as second reader and who developed the criteria presented in Section 7.

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1. Introduction

Since Admiral Farragut allegedly made the famous remark, "Damn the torpedoes, full speed ahead!" at Mobile during The War Between the States, the sea mine has evolved into a naval weapon of significance. Those Naval personnel assigned in Minesweepers and other Mine Warfare billets, and those few scientists who crusade for the cause in other branches of the Naval Establishment and in industry, probably realize and appreciate this statement more than anyone else in, or closely allied with, the Navy. It is common knowledge that the torpedoes alluded to were in actuallity contact mines of a very crude type. As is true with every weapon, the advent of modern warfare brought about a desire to improve existing weapons and to devise new and more ingenious ones. The trend has been, and rightly so, to build a bigger and better mouse trap. With modern advances comes complexity of construction and design, and with complexity, a great percentage of reliability is sacrificed. This sacrifice of reliability due to complexity in the mine cannot be tolerated, since reliability is considered to be the most sought after attribute in any weapons system.

The purpose of this paper is not to solve the problem of mine complexity vs. reliability in one quick stroke. It is proposed, however, that the mine designer may reduce the inherent complexity of the mine by looking more closely at the optimal sophistication that need be built into the mine.

The mine designer can design a mine with more or less sophistication as he sees fit. If the mine is highly sophisticated, it will accept targets of one size and that one alone. As the designed mine becomes less sophisticated it will accept targets in a narrow or wider range of sizes

depending upon the degree or amount of sophistication. In other words, a highly sophisticated mine is very selective, and as a result, manufacturing tolerances are small. Thus the mine complexity is increased. It is proposed that there exists an optimal sophistication that in a given situation, maximizes the probability that a mine is actuated by a target rather than by a minesweeper.

Let it be assumed in the problem at hand that a mine is designed with some particular ships signature in mind, even though the mine as actually manufactured will respond to a more or less wide range of signatures. Let this particular designed signature be referred to as the nominal signature of the mine.

Let the problem be split into two parts by assuming that the expression "percent deviation from nominal signature" has been given some operational definition and further that it can be measured. It may be rather difficult to accomplish this, though it is considered that, for large populations of mines, targets, and sweepers, some rough statistical definition will serve for practical purposes. The solution of this problem is not attempted.

Let the term "gate width" be used to define a term which is a measure of the deviation from nominal signature within which a given mine will respond to an actuation, and outside of which the mine will refuse to respond.

It is proposed to parameterize distributions of targets and sweepers signatures in terms of deviation from a nominal signature, i.e. "gate width," and to determine that "gate width" as a function of the parameters considered, that maximizes the chance that a mine is actuated by a target rather than by a minesweeper. This maximizing gate width then may be a guide to the mine designer as to the optimal sophistication needed in the mine in question.

It should be noted and emphasized that all the distributions of targets and sweepers considered are hypothetical and should not be construed to be actual ones. Some of the situations discussed are extreme cases and some could be considered trivial. It is felt, however, that actual distributions encountered in any given situation could possibly fall some place in between or be bracketed by those cases examined in the following sections. It should be further stressed that this study is not all inclusive. Only very small numbers of targets, sweepers, and minehunters are considered in the following cases because of the inordinate amount of calculations required to examine larger numbers of elements. It is believed that the consideration of larger more realistic numbers of targets, sweepers, and mine hunters is necessary as an extension of this study.

2. The General Probability Model

The purpose of this section is to present in general terms, those basic probability concepts that will be utilized in later sections to analyze specific models.

The principal probabilistic assumptions of this paper are the following:

(a) there are in operation n minesweepers and one target ship. At the time the mine becomes active any of the $(n + 1)!$ permutations of these ships passing the mine are equally likely;

(b) the signature of the target is a random variable with known probability density function $f_T(x)$;

(c) the signatures of the minesweepers are independent random variables with known probability density function $f_S(x)$.

A mine will be called successful if it actuates after being exposed to a target and unsuccessful if it is actuated by a sweeper. Therefore, the probability that a mine is successful can be given by P(T) = Probability (mine is actuated by the target).

Assumption (a) above implies that the target is equally likely to be in any position in the permutation of the n + 1 ships passing the mine, hence

$$(2.1)\quad P(T) = \frac{1}{n+1} P(TRS) + \frac{1}{n+1} P(TRS)\left[1-P(SRS)\right] + \cdots + \frac{1}{n+1} P(TRS)\left[1-P(SRS)\right]^{n},$$

where

$$(2.2)\quad P(TRS) = \int_{\mu-\gamma}^{\mu+\gamma} f_T(x)\,dx,$$

and

$$(2.3)\ P(SRS) = \int_{\mu-\gamma}^{\mu+\gamma} f_s(x)\, dx \quad .$$

P(TRS) and P(SRS) are the probabilities that the target and the sweeper are the "right size" respectively. μ is to be considered as the nominal signature and γ as one half the gate width with the assumption that the mine will be actuated in the interval $\mu \pm \gamma$. By recalling the formula for the partial sums of a geometric series, we may reduce (2.1) to

$$(2.4)\ P(T) = \frac{P(TRS)}{(n+1)\,P(SRS)} \left(1 - \left[1 - P(SRS)\right]^{n+1}\right)$$

The objective of this paper is to demonstrate which values of γ will maximize the effectiveness of the mine, i.e. maximize P(T). Sections 3 - 6 present numerical examples and analyses for some interesting hypothetical cases, while Section 7 presents a criterion for judging whether such numerical work is worthwhile in an individual situation.

3. Normally Distributed Target and Minesweeper Signatures.

In the first model considered, let it be assumed that the target traffic in a particular port is normally distributed with mean known μ and standard deviation σ . This distribution is illustrated in Fig. 1. If this is the situation, a mine used in this port would have a disigned nominal signature equal to the number μ . Because of manufacturing tolerances, etc., it is obvious that this mine will respond to actuations on either side of this nominal signature depending upon its sophistication. Since the target distribution is normal, the probability that a target at random will be of the right size, i.e. fall within a specified gate width, is equal to the area under the curve (Fig. 1) between $\mu - \gamma$ and $\mu + \gamma$ where γ may take on values from $-\infty$ to $+\infty$, i.e.

$$(3.1) \quad P(TRS) = \int_{\frac{\mu-\gamma}{\sigma}}^{\frac{\mu+\gamma}{\sigma}} \phi(x)\, dx ,$$

where

$$(3.2) \quad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

Let it also be assumed that the minesweeper signatures are normally distributed in the hypothetical port in question with mean μ and standard deviation $\tau\sigma$, as illustrated in Fig. 1 with $\tau = 1.6$. Here τ is defined as the constant which relates the standard deviation of the sweeper distribution to that of the target. As before, the probability that a sweeper will look to the mine as a ship within its gate width, is the area

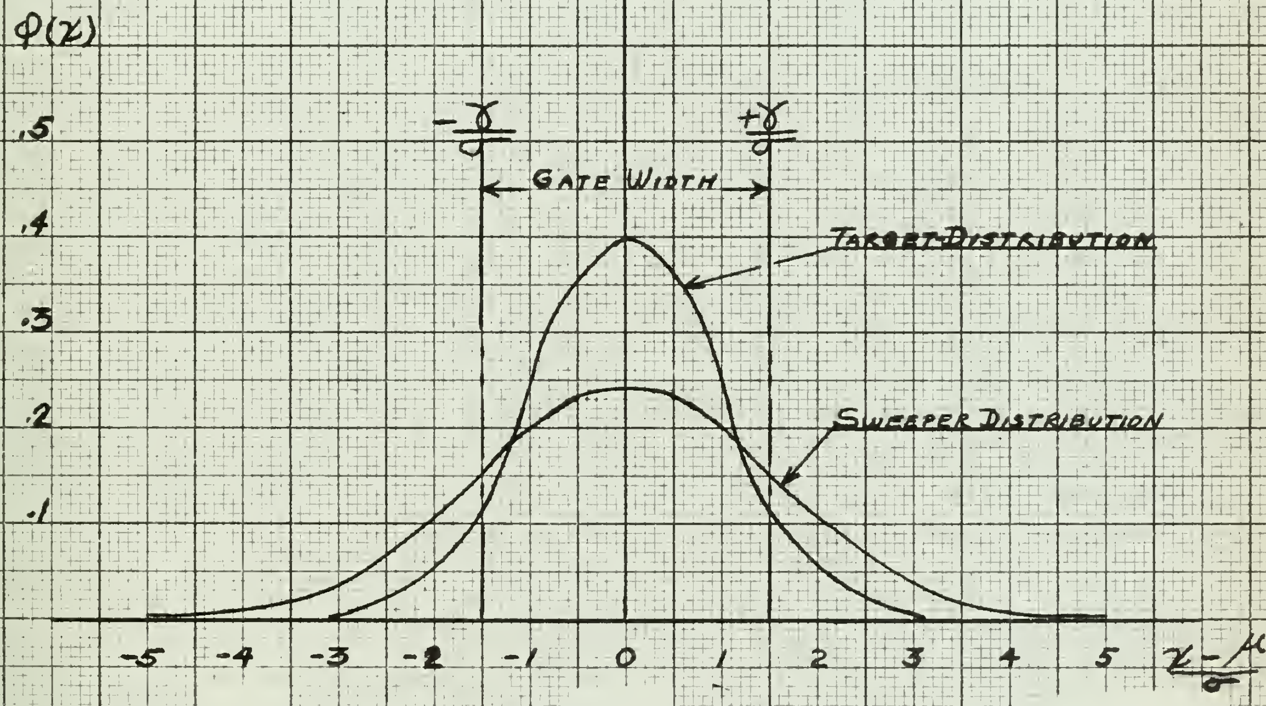

Figure 1. Normal Distribution of Targets and Sweepers

Illustration of Gate Width. (τ = 1.6)

under the sweeper distribution curve between $\mu - \gamma$ and $\mu + \gamma$. i.e.;

$$(3.3) \quad P(SRS) = \int_{\frac{\mu-\gamma}{\tau\sigma}}^{\frac{\mu+\gamma}{\tau\sigma}} \phi(x)\, dx,$$

where $\phi(x)$ is given by (3.2).

Values of the above defined probabilities are tabulated in Table 1 where $\tau = 1.6$.

In a situation where there are one sweeper and one target in operation, the probability that the mine will be actuated by the target and not by the sweeper, for a given gate width, is given by (2.1) where n = 1, as

(3.4) P(T) = P(TRS) x 1/2 + P(TRS) x 1/2 x [1-P(SRS)],

where, as before,

P(T) = Probability that the target will actuate the mine,

P(TRS) = Probability that the target is within the prescribed gate width (i.e. right size),

P(SRS) = Probability that the sweeper signature is within the prescribed gate width (i.e. right size).

An evaluation of (3.4) was made at a number of different gate widths and the results are tabulated in Table 1 and graphically displayed in Fig. 2 where P(T) is plotted versus half gate width. From Fig. 2 it is evident that a peak occurs at approximately $\frac{\gamma}{\sigma} = 1.8$, with the interpretation that this peak occurs at the optimum gate width, i.e. the gate width which maximizes the probability that the mine will be actuated by a target rather than by a minesweeper.

If we set $\mu = 5{,}000$ tons, $\sigma = 1{,}000$, and $\tau = 1.6$, a mine de-

$\frac{\gamma}{\sigma}$	P(TRS)	P(SRS)	P(T)
.2	.158	.098	.150
.4	.311	.196	.280
.6	.451	.292	.384
.8	.576	.380	.466
1.0	.682	.466	.523
1.2	.769	.546	.559
1.4	.838	.618	.579
1.5	.866	.648	.585
1.6	.890	.682	.586
1.7	.911	.710	.586
1.8	.928	.736	.587
1.9	.942	.764	.582
2.0	.954	.788	.578
2.2	.972	.828	.569
2.4	.983	.866	.557
2.6	.990	.894	.547
2.8	.995	.918	.538
3.0	.997	.938	.529
3.2	.998	.954	.522

$$P(T) = P(TRS) \times 1/2 + 1/2 \times P(TRS) \times [1-P(SRS)]$$

Table 1. Probabilities for Normal Target and Sweeper Distributions.
(τ = 1.6,1 Target and 1 Sweeper)

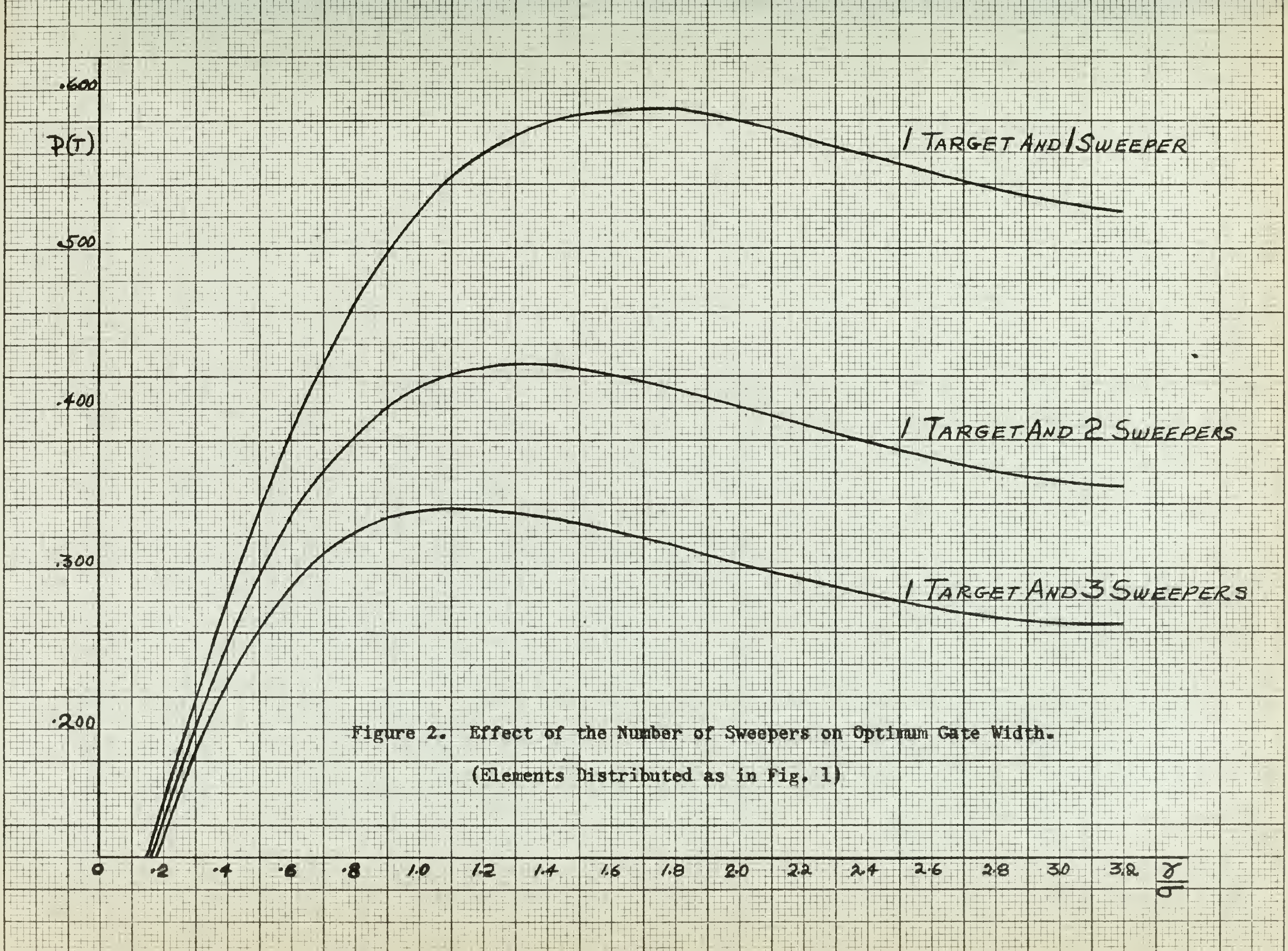

Figure 2. Effect of the Number of Sweepers on Optimum Gate Width.

(Elements Distributed as in Fig. 1)

signer should so sophisticate the mine that it accomodate targets between 3,200 tons and 6,800 tons. This design will result in a maximum probability P(T) of .587

The previously described methodology is next applied to a case where there is one target and two minesweepers. As before the distribution of the targets and the minesweepers is considered to be normal. In this situation, there are two sweepers and one target in operation. The probability that the mine will be actuated by the target and not by a sweeper for a given gate width is given by (2.1) where n = 2, as

$$(3.5)\ P(T) = P(TRS) \times 1/3 + [P(TRS) \times 1/3\ (1-P(SRS))] + (P(TRS) \times 1/3 \times [1-P(SRS)]^2).$$

An evaluation of (3.5) was made at a number of different gate widths and the results are tabulated in Table 2 and graphically displayed in Fig. 2. From the figure it is evident that the peak now occurs at $\frac{\gamma}{\sigma} = 1.3$ with the interpretation, as before, that this peak occurs at the optimum gate width.

If we now set μ = 5,000 tons, σ = 1,000, and τ = 1.6, it follows that a mine designer should, in the case of one target and two sweeper transits, so sophisticate the mine as to accomodate targets bebween 3,700 tons and 6,300 tons. This reduction in the optimum gate width appears logically correct in that it should be expected that when sweeper activity outweighs target activity, the gate width should be reduced to minimize the effectiveness of the sweepers. The above design will result in a maximum probability P(T) of .428.

A further increase in sweeper transits is next considered in the situation where there are three sweepers and one target. The previous analytical methods and assumptions apply, and the probability that a mine

$\frac{\gamma}{\sigma}$	P(T)
.2	.143
.4	.253
.6	.331
.8	.385
1.0	.413
1.2	.425
1.3	.428
1.4	.427
1.5	.425
1.6	.421
1.7	.417
1.8	.412
2.0	.399
2.2	.389
2.4	.377
2.6	.369
2.8	.361
3.0	.354
3.2	.348

$$P(T) = P(TRS) \times 1/3 + [P(TRS) \times 1/3 \times [1-P(SRS)]$$
$$+ (P(TRS) \times 1/3 \times [1-P(SRS)]^2)$$

Table 2. Values of P(T). (1 Target and 2 Sweepers Normally Distributed, τ = 1.6)

will be actuated by the target, and not by the sweeper, for a given gate width, is now given by (2.1) with n = 3, as

$$(3.6)\ P(T) = P(TRS) \times 1/4 + (P(TRS) \times 1/4 \times [1-P(SRS)]) + (P(TRS) \times 1/4 \times [1-P(SRS)]^2) + (P(TRS) \times 1/4 \times [1-P(SRS)]^3).$$

An evaluation of (3.6) was made at a number of different gate widths and the results are tabulated in Table 3 and graphically displayed in Fig. 2. From the figure it is evident that the peak now occurs at $\frac{\gamma}{\sigma} = 1.1$ with the interpretation as before that this peak occurs at the optimum gate width.

In the example where $\mu = 5{,}000$ tons, $\sigma = 1{,}000$, and $\tau = 1.6$, it follows that a mine designer should, in the case of one target and three sweeper transits, so sophisticate the mine that it accomodate targets between 3,900 tons and 6,100 tons. This design will result in a maximum probability, P(T) of .338.

As a further extension of the case wherein the target and the minesweepers are normally distributed, minehunting is introduced to determine the effect of it on the optimum sophistication. Minehunting may be differentiated from minesweeping in that it is primarily concerned with the location, physical removal, and/or destruction of the mine without regard to the mines influence mechanism. Minesweeping on the other hand, must actuate the mines influence mechanism before it can be successful. In as much as minehunting effort cannot be assigned a definite value, two arbitrary values, 50% and 33.3% of minehunting effectiveness were assumed in order to demonstrate a method of approach. It is further assumed in the evaluation of the minehunting effort that a minehunter will locate, remove or destroy every mine within the limits of the hunter's effectiveness.

$\frac{\gamma}{\sigma}$	P(T)
.2	.136
.4	.233
.6	.289
.8	.323
.9	.330
1.0	.336
1.1	.338
1.2	.337
1.4	.332
1.6	.323
1.8	.314
2.0	.302
2.2	.293
2.4	.284
2.6	.277
2.8	.270
3.0	.266
3.2	.265

$$P(T) = P(TRS) \times 1/4 + [P(TRS) \times 1/4 \times (1-P(SRS)]$$
$$+ (P(TRS) \times 1/4 \times [1-P(SRS)]^2)$$
$$+ (P(TRS) \times 1/4 \times [1-P(SRS)]^3)$$

Table 3. Values of P(T). (1 Target and 3 Sweepers Normally Distributed, $\tau = 1.6$)

In the situation where there is one target, one minesweeper, and one hunter (50% effective) transit all of which are equally likely events, the probability that a mine will be actuated by the target and not swept by the sweeper or found and removed by the hunter for a given gate width is given by:

$$(3.7)\ P(T) = 1/3 \times P(TRS) + [1/3 \times [1-P(SRS)] + 1/6][1/2 \times P(TRS)] + [1/6 \times [1-P(SRS)] \times P(TRS)]$$

Literally (3.7) states:

The probability that a target actuates the mine is equal to the probability that the target is of the right size and no sweeper or hunter precedes it; or, the target is of the right size and a sweeper or a hunter precedes it and is of the wrong size and misses; or, the target is of the right size and a hunter and a sweeper precede it and are of the wrong size and miss.

The tabulated values of (3.7) are contained in Table 4 and graphically displayed in Fig. 3. From the figure it is evident that the peak occurs at $\frac{\gamma}{\sigma}$ = 1.8, the optimum gate width previously obtained for the 1 target and 1 sweeper case, with a decrease in the maximum P(T) to .468. This is logically acceptable in that it might be expected that the presence of a hunter would only reduce the over all probability of the target actuating the mine, because of the obvious reduction in the mine population.

In like manner, a hunter that is 33.3% effective will also effect the mine population but not as much as one that is 50% effective. P(T) can be written as follows:

$$(3.8)\ P(T) = 1/3 \times P(TRS) + [1/3 \times [1-P(SRS)] + 2/9][1/2 \times P(TRS)] + [2/9 \times [1-P(SRS)] \times P(TRS)].$$

$\frac{\gamma}{\sigma}$	M/H 50% Effective P(T)	M/H 33.3% Effective P(T)
.2	.113	.125
.4	.213	.235
.6	.294	.324
.8	.358	.395
1.0	.405	.444
1.2	.436	.477
1.4	.455	.496
1.5	.462	.503
1.6	.465	.506
1.7	.467	.507
1.8	.468	.508
1.9	.466	.505
2.0	.464	.502
2.2	.460	.496
2.4	.453	.488
2.6	.447	.480
2.8	.441	.474
3.0	.435	.467
3.2	.430	.461

Table 4. Values of P(T). (1 Target, 1 Sweeper, and 1 Minehunter, τ = 1.6)

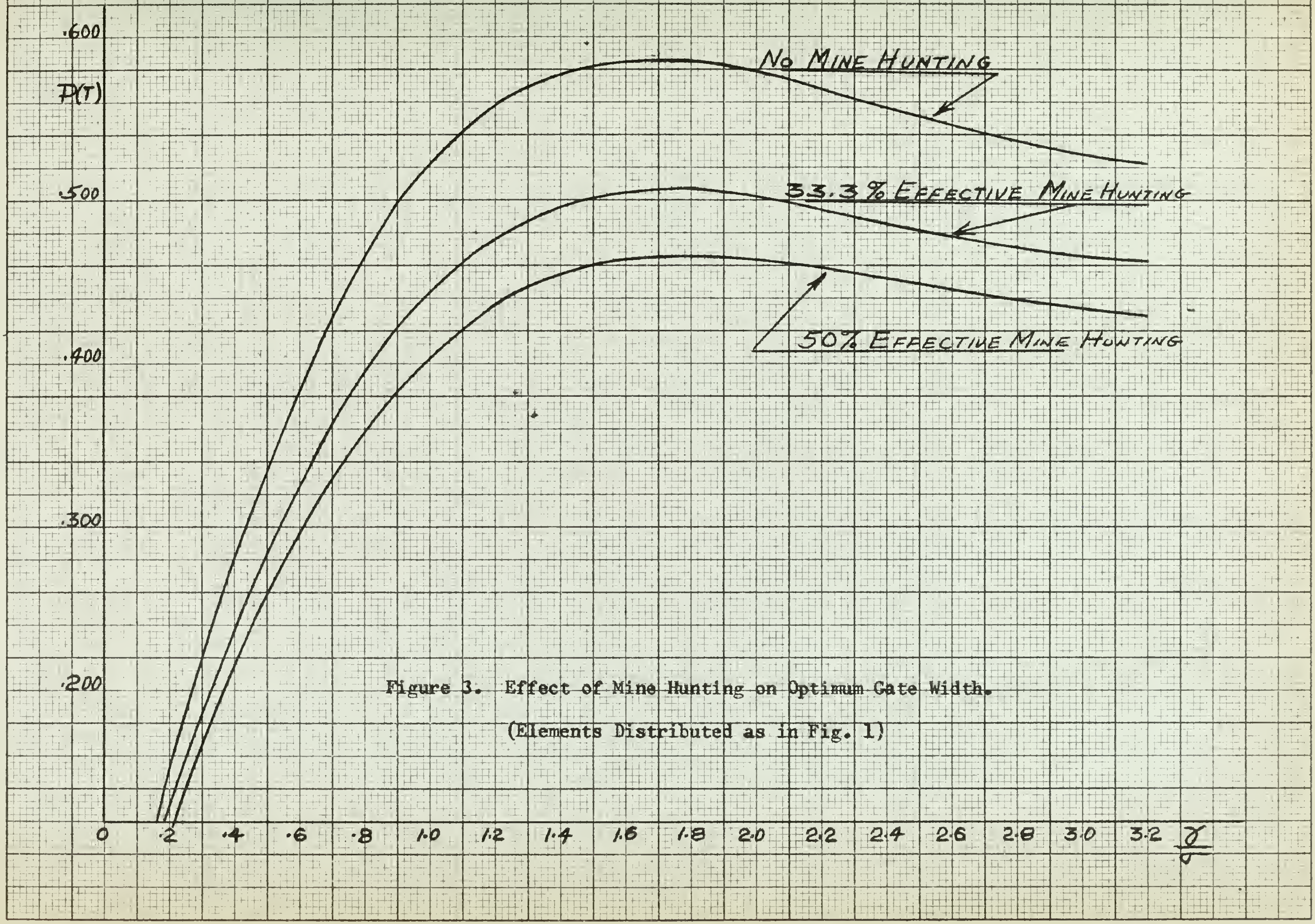

Figure 3. Effect of Mine Hunting on Optimum Gate Width.

(Elements Distributed as in Fig. 1)

An evaluation of (3.8) is tabulated in Table 4 and graphically displayed in Fig. 3. As intuitively expected, the curve peak is again at $\frac{\gamma}{\sigma}$ = 1.8 and lies between the no minehunting curve and the 50% effective curve. It is obvious at this point that minehunting does not effect the optimum sophistication of a mine, but does effect the overall P(T). This statement will be examined further in cases yet to be discussed.

As a further extension of the normal distribution case, the two distributions previously considered were interchanged, i.e., the higher more peaked one in Fig. 1 is now considered the sweeper distribution and the other the target distribution. An evaluation of (3.4) in this case yields the values shown in Table 5 and are graphically displayed in Fig. 4. Here no peak is observed and it is therefore evident that certain criteria must be established before a peak will be obtained. Such criteria will be examined in Section 7.

$\frac{\gamma}{\sigma}$	P(T)
.2	.090
.6	.226
1.0	.307
1.4	.359
1.6	.378
1.8	.394
2.0	.412
2.2	.426
2.4	.440
2.6	.451
2.8	.461
3.0	.470
3.2	.478

Table 5. Values of P(T). (1 Target and 1 Sweeper Normally Distributed, $\tau = \frac{1}{1.6} = .625$)

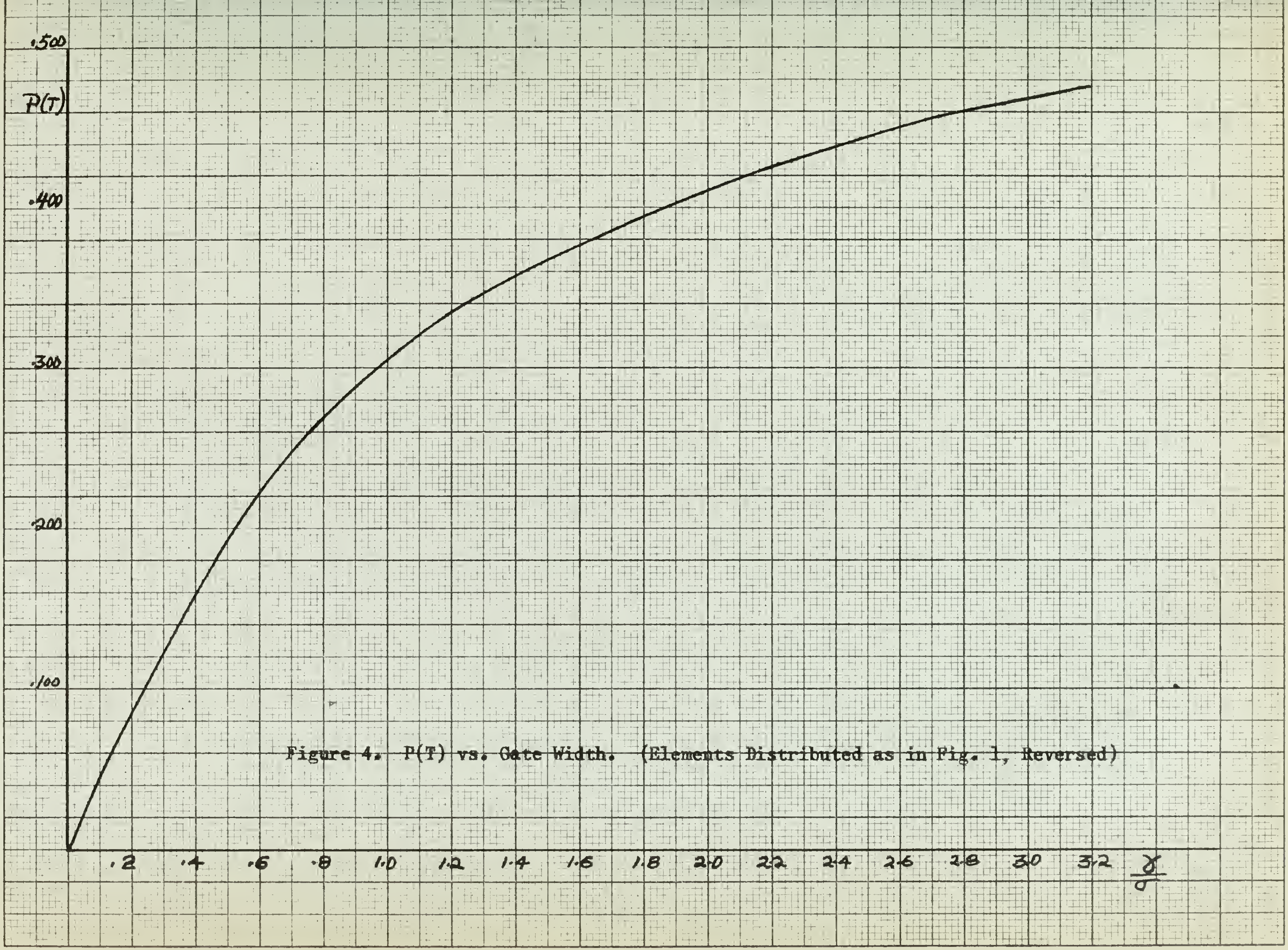

Figure 4. P(T) vs. Gate Width. (Elements Distributed as in Fig. 1, Reversed)

Summary:

Before proceeding with the analysis of other distributions of targets and sweepers, a summary of previous observations seems appropriate.

1. When a small number of targets and sweepers is normally distributed as illustrated in Fig. 1, and when the target standard deviation is smaller than the sweeper standard deviation, an optimum gate width or optimum sophistication exists.
2. With one target, an increase in the number of sweepers reduces the optimum gate width.
3. The introduction of minehunting does not effect the optimum gate width, but does reduce the over all probability that the target will actuate the mine.

4. A Two Peak Target Distribution With a Triangular Sweeper Distribution.

The next model to be considered is a very special numerical example in which it is assumed that the target traffic is continuously distributed as illustrated in Fig. 5 with a μ of 15,000 tons. In this case, the probability that a target at random will be of the right size is equal to the area under the normalized target distribution curve (Fig. 5) within the prescribed gate width. The measurement of these areas was done by planimeter and the values of the probabilities are tabulated in Table 6.

The sweeper distribution curve in this model is a triangular one, also illustrated in Fig. 5. The probability then, as before, that a sweeper will look to a mine as a target within its gate width is the area under the normalized sweeper distribution curve within the prescribed gate width.

In a situation where there is one sweeper and one target in operation, the probability that the mine will be actuated by the target and not by the sweeper for a given gate width is, as before

(4.1) $P(T) = 1/2 \times P(TRS) + 1/2 \times P(TRS) \times [1-P(SRS)]$.

The results of an evaluation of (4.1) at a number of different gate widths are tabulated in Table 6 and graphically displayed in Fig. 6. From Fig. 6, it is evident that a peak occurs at 11,000 tons with the interpretation that this is the gate width which maximizes the probability that the mine will be actuated by the target rather than by a minesweeper. It follows, therefore, that a mine designer in this case should so sophisticate his mine that it accomodate targets between 4,000 tons and 26,000 tons. This design will result in a maximum probability P(T) of .581.

A similar analysis is next applied to the case when there is one target and two minesweepers distributed as before. In this case, the probability that the mine will be actuated by the target and not by a

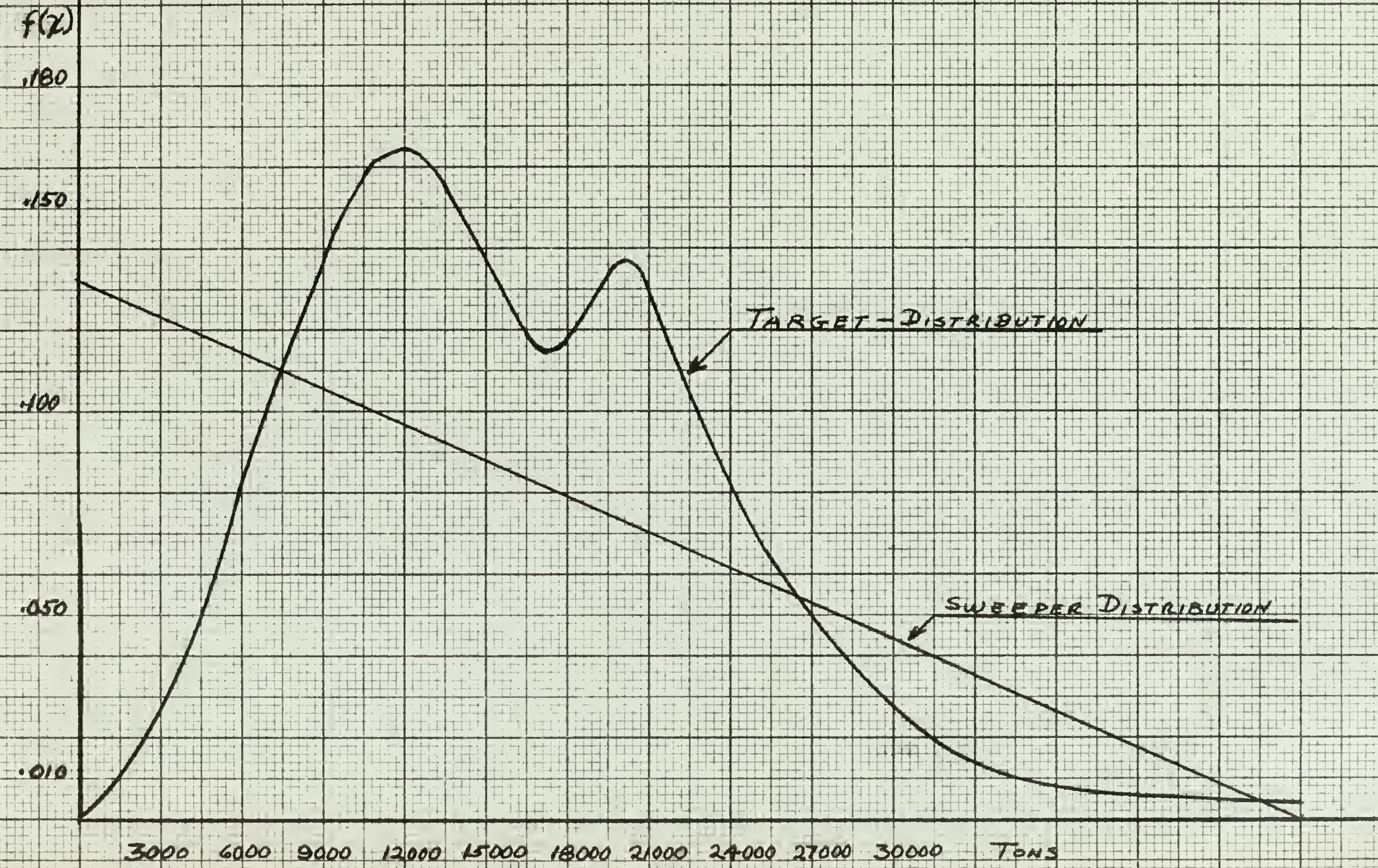

Figure 5. Two Peak Target Distribution and Triangular Sweeper Distribution.

γ	P(TRS)	P(SRS)	P(T)
1000	.092	.059	.089
2000	.184	.119	.173
3000	.275	.179	.250
4000	.371	.238	.326
5000	.468	.295	.399
6000	.562	.357	.462
7000	.644	.415	.510
8000	.715	.472	.546
9000	.773	.535	.566
10000	.825	.595	.579
11000	.864	.655	.581
12000	.890	.713	.573
13000	.912	.770	.561
14000	.930	.830	.544
15000	.945	.890	.524

Table 6. Probabilities for Two Peak Target and Triangular Sweeper Distribution. (1 Target and 1 Sweeper Distributed as Illustrated in Fig. 5)

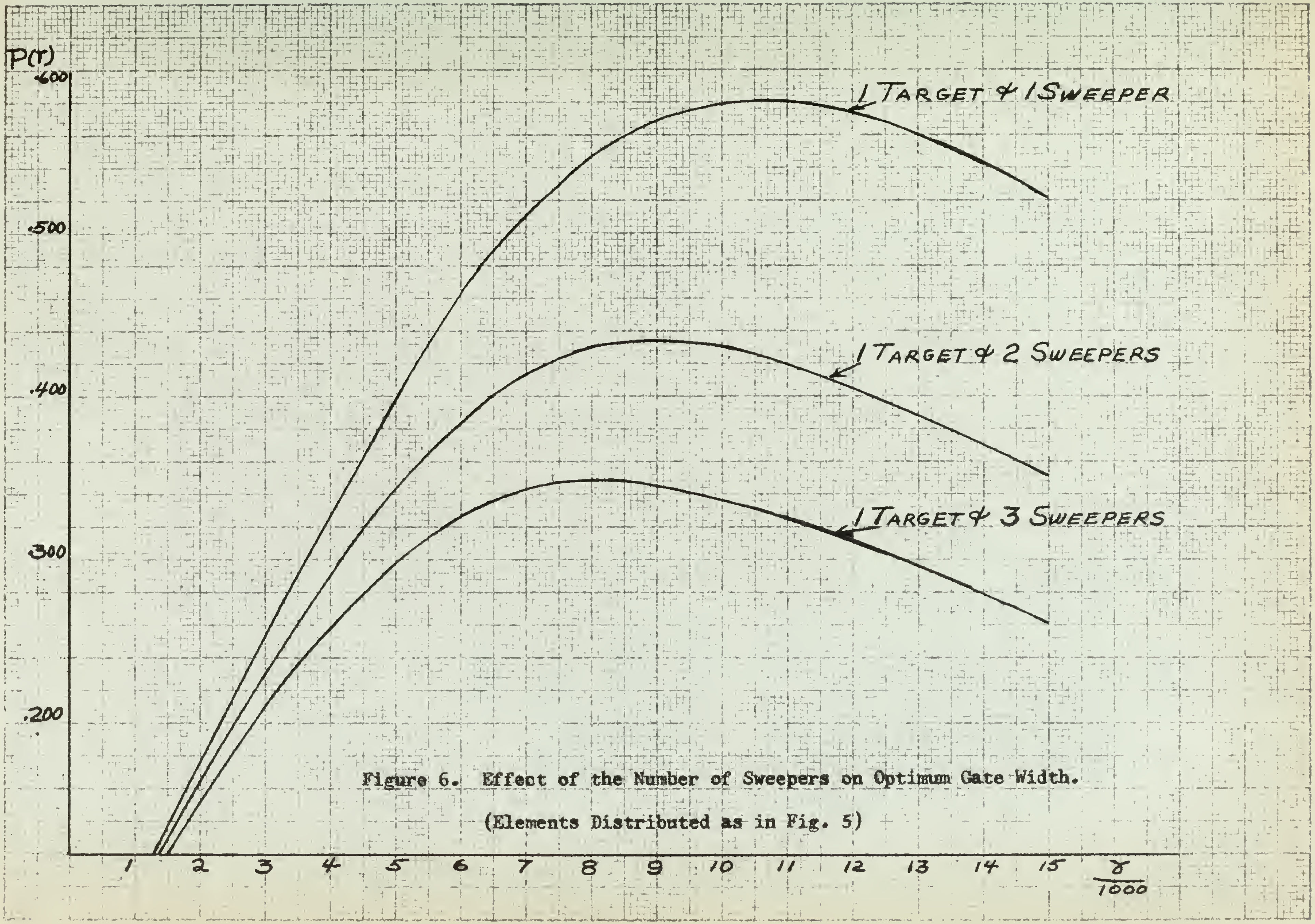

Figure 6. Effect of the Number of Sweepers on Optimum Gate Width.

(Elements Distributed as in Fig. 5)

sweeper for a given gate width is given by

$$(4.2)\ P(T) = 1/3 \times P(TRS) + [1/3 \times P(TRS) \times (1-P(SRS))] + [1/3 \times P(TRS) \times [1-P(SRS)]^2].$$

An evaluation of (4.2) at a number of different gate widths is tabulated in Table 7 and graphically displayed in Fig. 6. The peak now occurs at 9,000 tons with the interpretation, as before, that this peak occurs at the optimum gate width. It follows that the mine should now be designed to accomodate targets between 6,000 tons and 24,000 tons. This reduction in the optimum gate width, as in the case of the situation discussed in Section 3, is necessary to minimize the effectiveness of the increased sweeper activity. The above design will result in a maximum probability P(T) of .433.

A further increase in sweeper transits is next considered in the situation where there are three sweepers and one target. The previous analytical methods and assumptions apply, and the probability that a mine will be actuated by the target and not by a sweeper for a given gate width, is now given by

$$(4.3)\ P(T) = 1/4 \times P(TRS) + (1/4 \times P(TRS) \times [1-P(SRS)]) + (1/4 \times P(TRS) \times [1-P(SRS)]^2) + (1/4 \times P(TRS) \times [1-P(SRS)]^3).$$

An evaluation of (4.3) was made at a number of different gate widths and the results are tabulated in Table 7 and graphically displayed in Fig. 6. From the figure it is evident that the peak now occurs at 8,000 tons with the interpretation as before that this peak occurs at the optimum gate width. It follows now that the mine should be designed to accept targets between 7,000 tons and 23,000 tons. This design will result in a maximum probability P(T) of .350.

γ	P(T) 1 Target, 2 Sweepers	P(T) 1 Target, 3 Sweepers
1000	.086	.084
2000	.163	.154
3000	.229	.210
4000	.289	.258
5000	.344	.299
6000	.385	.326
7000	.414	.342
8000	.431	.350
9000	.433	.344
10000	.432	.337
11000	.422	.325
12000	.406	.310
13000	.390	.295
14000	.372	.280
15000	.353	.265

Table 7. Values of P(T). (1 Target and 2,3 Sweepers Distributed as Illustrated in Fig. 5)

As a further extension of the case wherein the targets and the minesweepers are distributed as illustrated in Fig. 5, minehunting is introduced to determine the effect of it on the optimum sophistication. As in Section 3, arbitrary values of minehunting effectiveness are assumed. In this case three values were chose; 10%, 50%, and 90%, which bracket any expected values that minehunting effectiveness might assume.

In the situation, then, where there is one target, one minesweeper, and one hunter transit all of which are equally likely events, the probability that a mine will be actuated by the target and not swept by the sweeper or found and removed by the hunter for a given gate width is given by

Minehunting 10% effective:

$$(4.4)\quad P(T) = 1/3 \times P(TRS) + [1/3 \times [1-P(SRS)] + .3][1/2 \times P(TRS)] + [.3 \times [1-P(SRS)] \times P(TRS)]$$

Minehunting 50% effective:

$$(4.5)\quad P(T) = 1/3 \times P(TRS) + [1/3 \times [1-P(SRS)] + 1/6][1/2 \times P(TRS)] + [1/6 \times [1-P(SRS)] \times P(TRS)]$$

Minehunting 90% effective:

$$(4.6)\quad P(T) = 1/3 \times P(TRS) + [1/3 \times [1-P(SRS)] + 1/30][1/2 \times P(TRS)] + [1/30 \times [1-P(SRS)] \times P(TRS)]$$

The tabulated values of (4.4), (4.5), and (4.6) are contained in Table 8 and graphically displayed in Fig. 7. From the figure it is evident that the peak of each curve occurs at 11,000 tons, the optimum gate width previously obtained for the one target and one sweeper case, with a corresponding decrease in P(T). This stability of optimum gate width and the decrease in P(T) was also observed when the targets and sweepers were normally distributed. The statement previously made, that mine-

γ	M/H 10% Effective	M/H 50% Effective	M/H 90% Effective
1000	.085	.067	.049
2000	.164	.131	.097
3000	.238	.190	.141
4000	.311	.249	.186
5000	.380	.305	.230
6000	.440	.354	.269
7000	.487	.394	.301
8000	.521	.423	.325
9000	.542	.442	.342
10000	.554	.456	.356
11000	.557	.459	.362
12000	.549	.456	.361
13000	.538	.452	.361
14000	.523	.440	.357
15000	.505	.428	.351

Table 8. Values of P(T). (1 Target, 1 Sweeper, and 1 Minehunter)

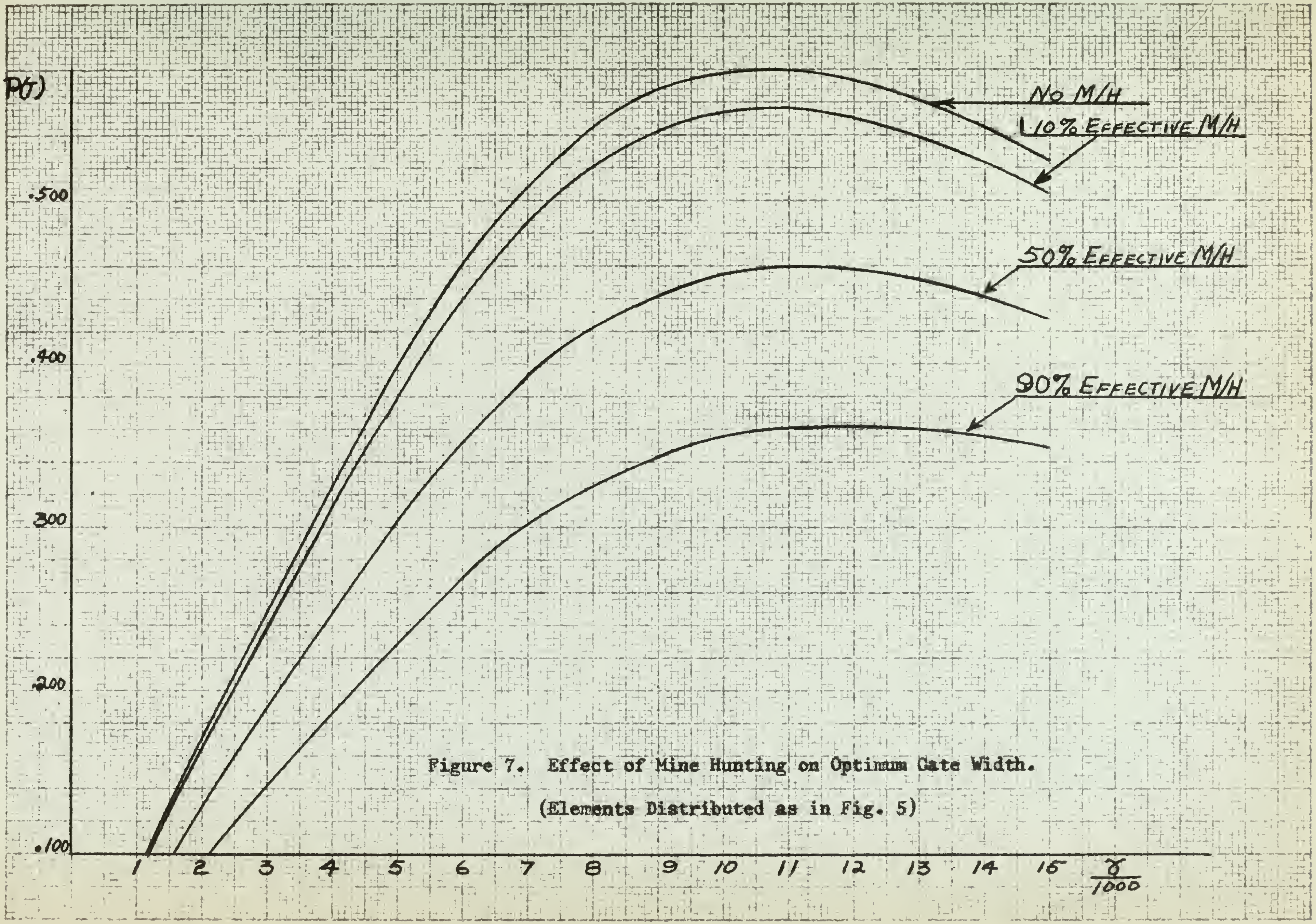

Figure 7. Effect of Mine Hunting on Optimum Gate Width.

(Elements Distributed as in Fig. 5)

hunting does not affect the optimum sophistication of a mine, but does affect the over all P(T) is obviously true in the instant case.

In order to further investigate the nature of the effect of minehunting on the over all probability P(T), minesweepers were eliminated from the previous case. In this model P(T) was calculated when there is no minesweeping and with minehunting which is 0%, 10%, 50% and 90% effective.

In the case where there is

No minesweeping and 0% effective minehunting,

(4.7) $P(T) = P(TRS)$

No minesweeping and 10% effective minehunting,

(4.8) $P(T) = 1/2 \times P(TRS) + [1/2 \times P(TRS) \times .9]$

No minesweeping and 50% effective minehunting,

(4.9) $P(T) = 1/2 \times P(TRS) + [1/2 \times P(TRS) \times .5]$

No minesweeping and 90% effective minehunting,

(4.10) $P(T) = 1/2 \times P(TRS) + [1/2 \times P(TRS) \times .1]$.

In each case above, P(T) is an increasing function of P(TRS) alone and hence of γ. This implies that the maximizing value of γ is ∞. An increase in mine population would, however, be needed to counteract increased minehunting effectiveness.

5. A Normal Target Distribution With a Triangular Sweeper Distribution.

In the next numerical example to be considered, let it be assumed that the target traffic is normally distributed as illustrated in Fig. 8 with mean μ of 5,000 tons and standard deviation σ of 1,000. The sweeper distribution is again assumed to be triangular, also illustrated in Fig. 8. This model, a slight variation of those previously discussed, is examined to again observe the nature of optimum gate width and its dependence on the target distribution and the sweeper distribution.

The values of P(TRS), P(SRS), and P(T) for one target and one sweeper are contained in Table 9, and graphically displayed in Fig. 9. From the figure it is evident that a peak, interpreted as the optimum gate width, occurs at $\frac{\gamma}{\sigma} = 1.9$.

With μ = 5,000 tons and σ = 1,000 a mine designer should so sophisticate the mine that it accomodate targets between 3,100 tons and 6,900 tons. This design will result in a maximum probability P(T) of .769.

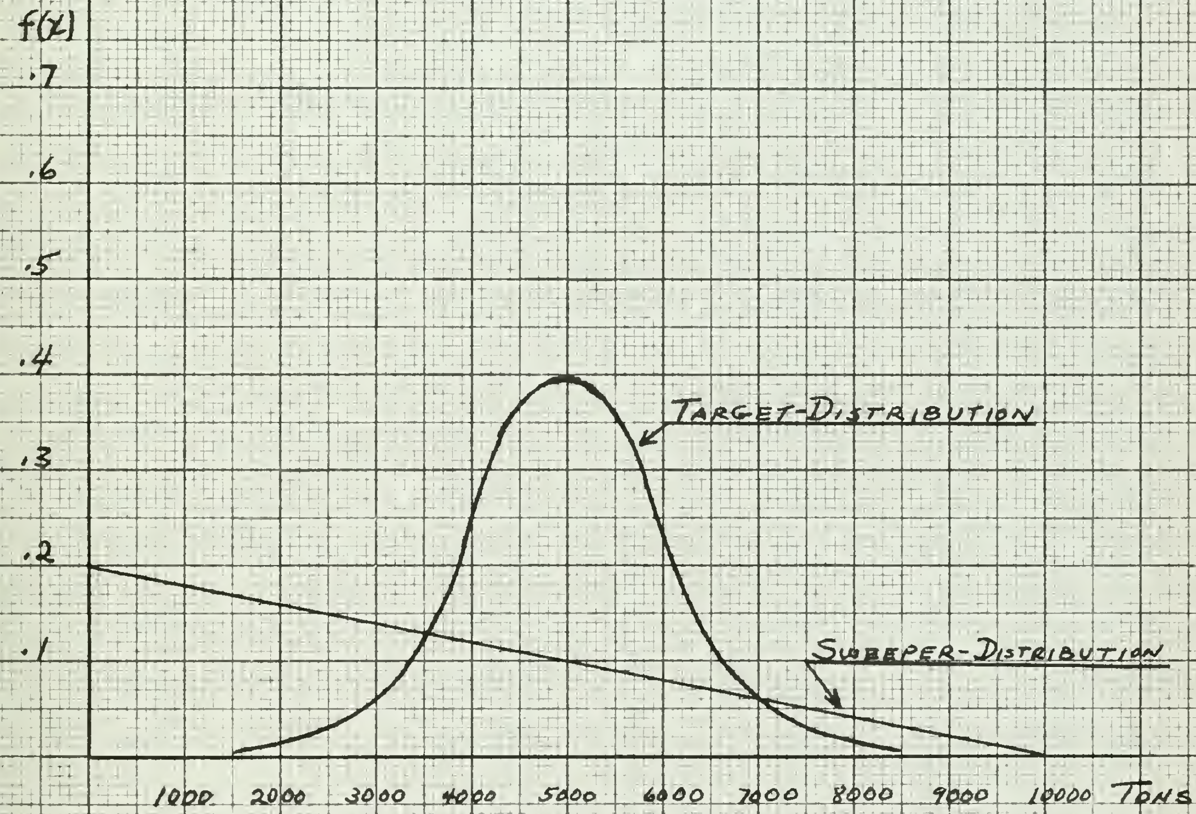

Figure 8. Normal Target Distribution and Triangular Sweeper Distribution (σ = 1000)

γ	P(TRS)	P(SRS)	P(T)
200	.158	.039	.155
400	.311	.078	.299
600	.451	.116	.424
800	.576	.156	.531
1000	.682	.194	.616
1200	.769	.232	.679
1400	.838	.272	.724
1500	.866	.291	.740
1600	.890	.312	.751
1700	.911	.330	.760
1800	.928	.351	.765
1900	.942	.369	.769
2000	.954	.390	.768
2200	.972	.427	.765
2400	.983	.468	.753
2600	.990	.509	.738
2800	.995	.550	.721
3000	.997	.589	.699
3200	.998	.628	.684

Table 9. Probabilities for Normal Target and Triangular Sweeper Distribution. (1 Target and 1 Sweeper, σ = 1000)

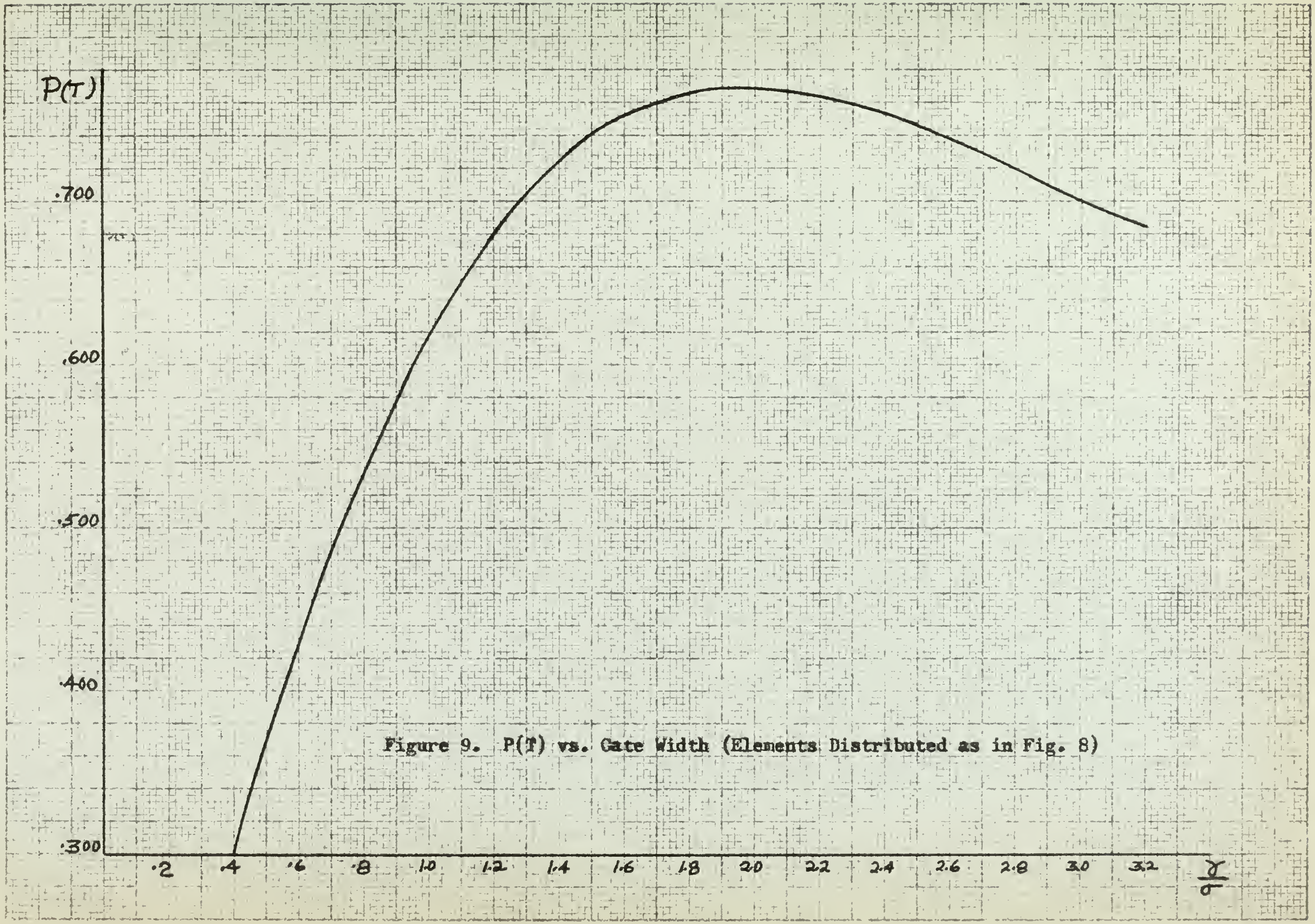

Figure 9. P(T) vs. Gate Width (Elements Distributed as in Fig. 8)

6. A Two Peak Sweeper Distribution and a One Peak Target Distribution.

The last numerical example to be examined is an extreme and most unlikely one. It is chosen to further investigate the nature of the dependence of optimum gate width on the target and sweeper densities. Let it be assumed that the targets and the sweepers are continuously distributed as illustrated in Fig. 10 with a μ = 15,000 tons.

An evaluation of P(T) at a number of different gate widths was made and the results are contained in Table 10 and graphically displayed in Fig. 11. It is evident that a peak occurs at 3,000 tons, the optimum gate width. This implies that a mine designer should sophisticate the mine to accomodate targets between 12,000 tons and 18,000 tons. Note that a finite value is again obtained for the optimum gate width.

As a further extension of this case, the two distributions are interchanged, i.e., the one peak curve is now considered the sweeper distribution curve and the other, the target distribution curve. An evaluation of P(T) in this case yields the values shown in Table 11 and is graphically displayed in Fig. 12. Here a peak is evident at about 11,000 tons. This implies that a mine designer should design the mine to accomodate targets between 4,000 tons and 26,000 tons.

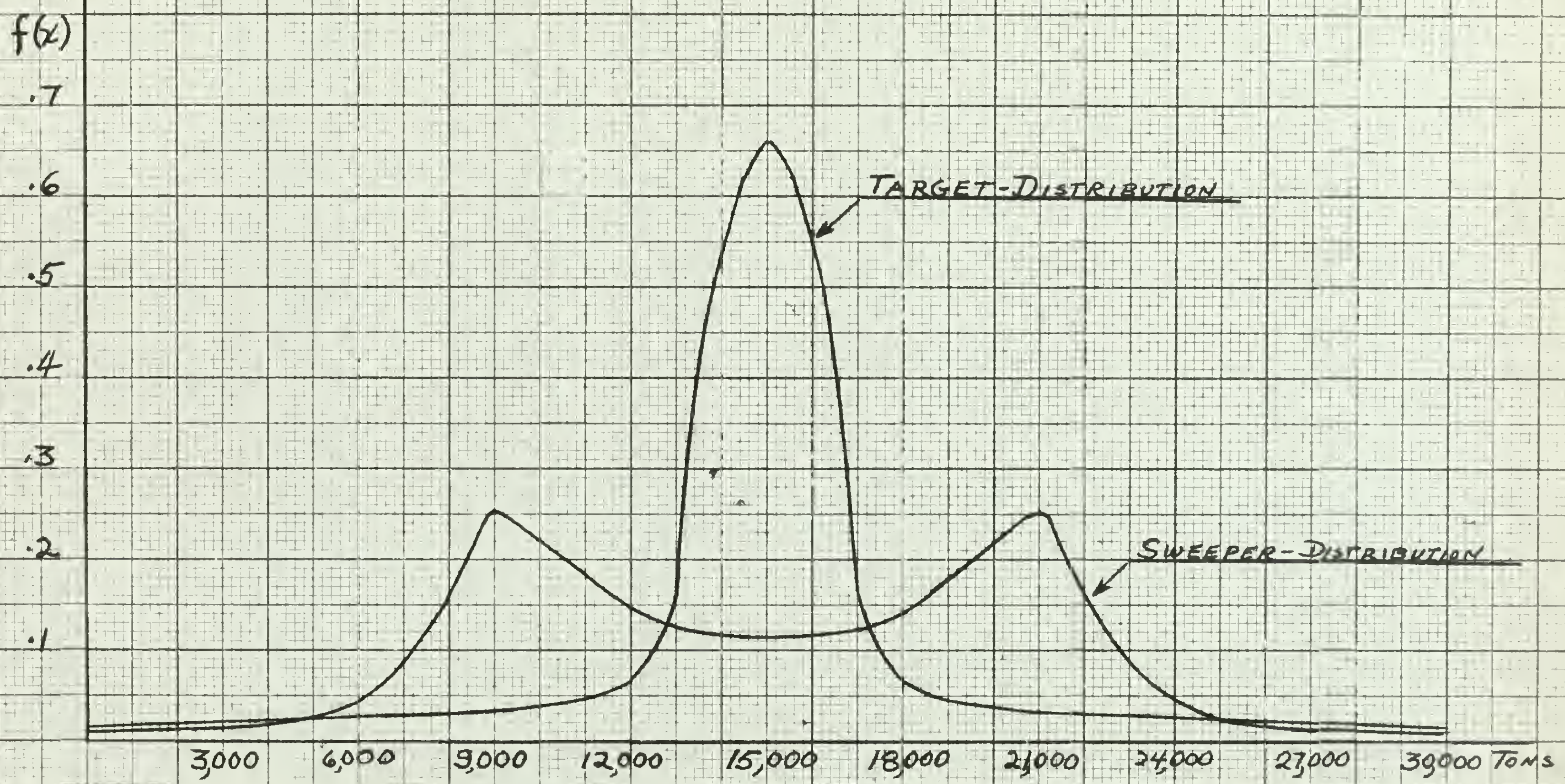

Figure 10. Two Peak Sweeper Distribution and One Peak Target Distribution.

γ	P(TRS)	P(SRS)	P(T)
1000	.413	.078	.397
2000	.668	.161	.614
3000	.740	.252	.647
4000	.784	.398	.628
5000	.811	.499	.608
6000	.838	.662	.560
7000	.866	.806	.517
8000	.884	.880	.495
9000	.907	.920	.490
10000	.921	.941	.493
11000	.940	.959	.489
12000	.957	.970	.493
13000	.970	.981	.494
14000	.990	.990	.500

Table 10. Probabilities for Two Peak Sweeper and One Peak Target Distribution. (1 Target and 1 Sweeper Distributed as in Fig. 10)

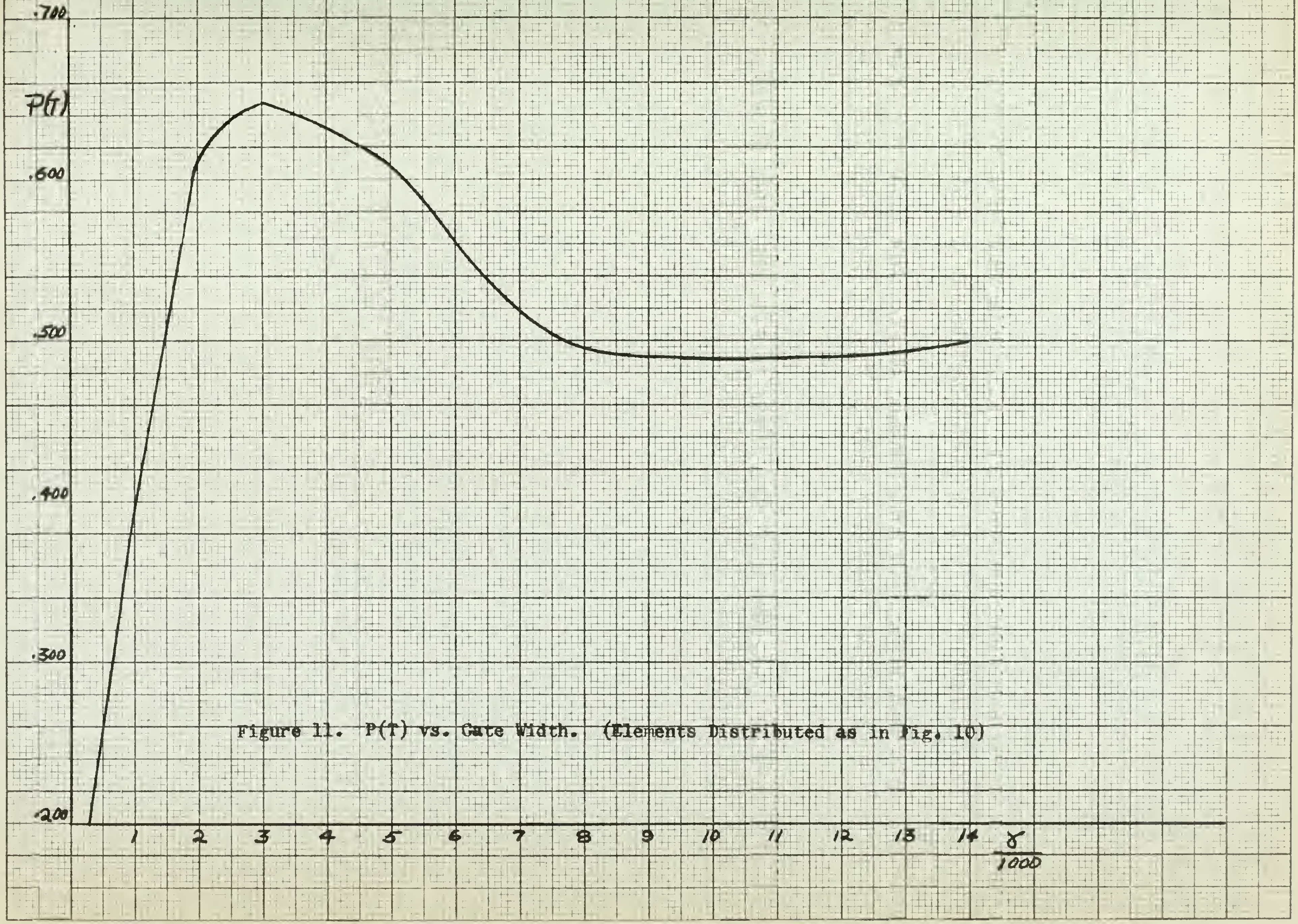

Figure 11. P(T) vs. Gate Width. (Elements Distributed as in Fig. 10)

γ	P(TRS)	P(SRS)	P(T)
1000	.078	.413	.062
2000	.161	.668	.107
3000	.252	.740	.159
4000	.398	.784	.242
5000	.499	.811	.297
6000	.662	.838	.385
7000	.806	.866	.457
8000	.880	.884	.491
9000	.920	.907	.503
10000	.941	.921	.507
11000	.959	.940	.509
12000	.970	.951	.505
13000	.981	.970	.5047
14000	.990	.990	.500

Table 11. Probabilities for Two Peak Target and a One Peak Sweeper Distribution. (1 Target and 1 Sweeper)

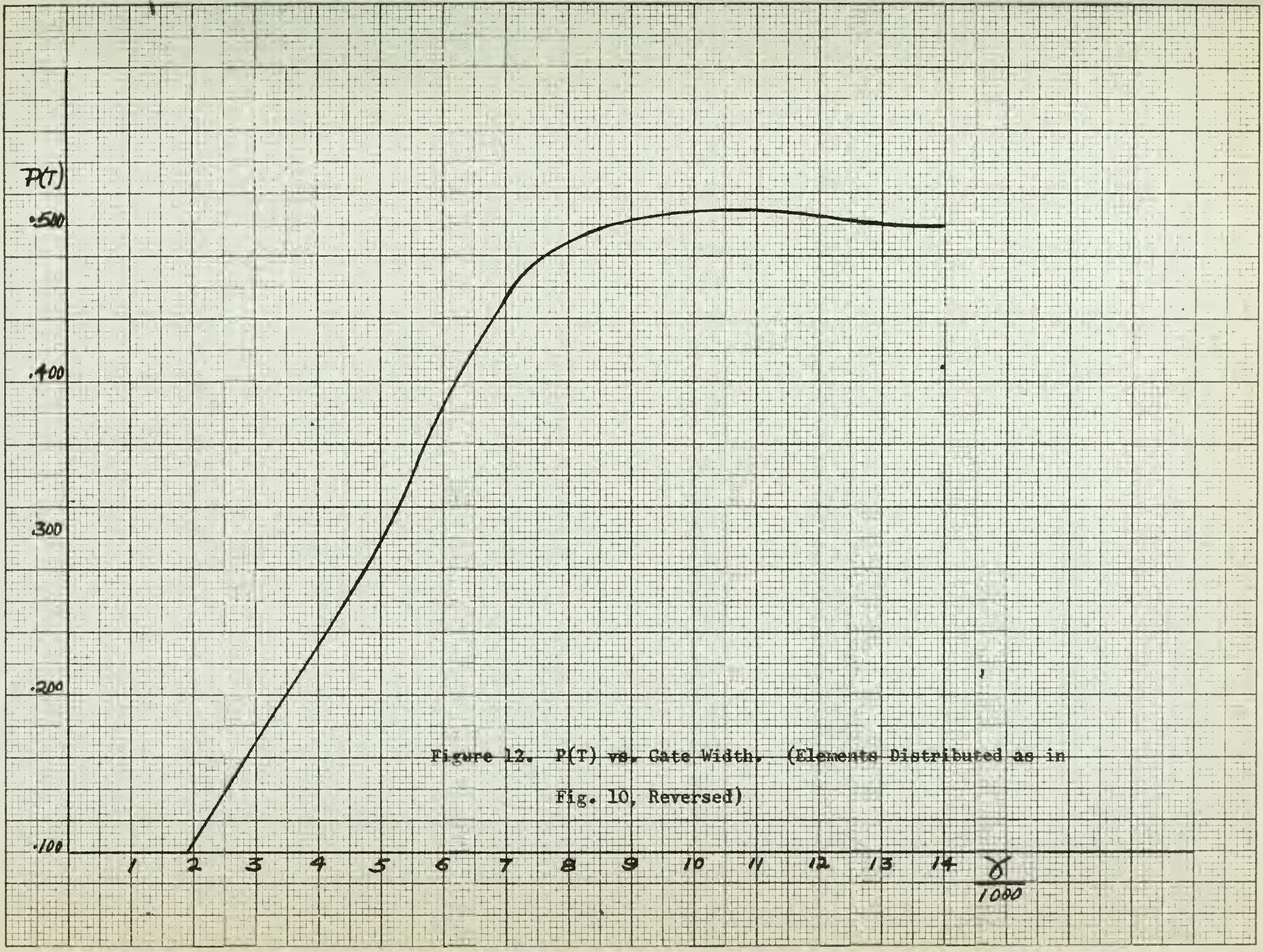

Figure 12. P(T) vs. Gate Width. (Elements Distributed as in Fig. 10, Reversed)

7. Some Remarks Concerning the General Case.

It was shown in Section 2 (c.f. (2.4)) that:

$$(7.1)\quad P(T) = \frac{P(TRS)}{(n+1)P(SRS)}\left[1-(1-P(SRS))^{n+1}\right]$$

It is clear that if P(T) is regarded as a function of γ , then:

$$(7.2)\quad \lim_{\gamma \to \infty} P(T) = \frac{1}{n+1}$$

so that if there is a value γ_0 such that P(T) $> \frac{1}{n+1}$ for $\gamma = \gamma_0$ there will be a finite optimal gate width. A sufficient condition that there be such a γ_0 is that:

$$(7.3)\quad \frac{\partial P(T)}{\partial \gamma} < 0 \qquad \text{for all sufficiently large } \gamma .$$

Now:

$$(7.4)\quad \frac{\partial P(T)}{\partial \gamma} = \frac{1}{n+1}\left[\frac{\frac{\partial P(TRS)}{\partial \gamma}}{P(SRS)} - \frac{P(TRS)\frac{\partial P(SRS)}{\partial \gamma}}{P(SRS)^2}\right]$$

$$\times\left[1-(1-P(SRS))^{n+1}\right]$$

$$+\frac{P(TRS)}{P(SRS)}\;\frac{\partial P(SRS)}{\partial \gamma}\left(1-P(SRS)\right)^{n}$$

Dividing by $\frac{\partial P(SRS)}{\partial \gamma}$ and transposing, we may express the condition (7.3) as:

$$(7.5)\quad \frac{\frac{\partial P(TRS)}{\partial \gamma}}{\frac{\partial P(SRS)}{\partial \gamma}} < \frac{P(TRS)}{P(SRS)} - (n+1)P(TRS)\left[1-P(SRS)\right]^{n}$$

Recalling from (2.2) and (2.3) that P(TRS) and P(SRS) tend to unity as γ becomes large we now see that a sufficient condition that P(T) have a maximum is that:

$$\text{(7.6)} \quad \lim_{\gamma \to \infty} \frac{\dfrac{\partial P(TRS)}{\partial \gamma}}{\dfrac{\partial P(SRS)}{\partial \gamma}} < 1$$

One interesting example is that of Section 3. With P(TRS) and P(SRS) given by (3.1) and (3.3) respectively we have:

$$\text{(7.7)} \quad \frac{\partial P(TRS)}{\partial \gamma} = \frac{1}{\sqrt{2\pi}\,\sigma}\left(e^{\frac{-(\mu+\gamma)^2}{2\sigma^2}} + e^{\frac{-(\mu-\gamma)^2}{2\sigma^2}}\right),$$

and

$$\text{(7.8)} \quad \frac{\partial P(SRS)}{\partial \gamma} = \frac{1}{\sqrt{2\pi}\,\tau\sigma}\left(e^{\frac{-(\mu+\gamma)^2}{2\tau^2\sigma^2}} + e^{\frac{-(\mu-\gamma)^2}{2\tau^2\sigma^2}}\right)$$

Now

$$\text{(7.9)} \quad \frac{\dfrac{\partial P(TRS)}{\partial \gamma}}{\dfrac{\partial P(SRS)}{\partial \gamma}} = \tau\, e^{\frac{\mu^2+\gamma^2}{2}\left(\frac{1}{\tau^2\sigma^2} - \frac{1}{\sigma^2}\right)} \left(\frac{\cosh\left(\frac{\mu\gamma}{\sigma^2}\right)}{\cosh\left(\frac{\mu\gamma}{\tau^2\sigma^2}\right)}\right)$$

and:

$$\text{(7.10)} \quad \lim_{\gamma \to \infty} \frac{\dfrac{\partial P(TRS)}{\partial \gamma}}{\dfrac{\partial P(SRS)}{\partial \gamma}} = \begin{cases} 0, & \tau > 1 \\ 1, & \tau = 1 \\ \infty, & \tau < 1 \end{cases}$$

so that (7.3) is satisfied if and only if $\tau > 1$, i.e. the standard deviation of the sweeper distribution is greater than that of the target distribution. This is, of course, in accord with the numerical findings of Section 3.

The evaluation of the optimal gate width by the methods of this section seems to be impossible since it would involve solving the equation

$$(7.11) \quad \frac{\partial P(T)}{\partial \gamma} = 0$$

where $\frac{\partial P(T)}{\partial \gamma}$ is given by (7.4). Even in the examples of Section 3 with $n = 1$ this calculation seems quite difficult.

Furthermore, it should be noted that (7.6) provides only a sufficient condition that P(T) have a maximum for some $\gamma < \infty$. There is no necessary condition available except that $f_T(x)$ and $f_S(x)$ are continuous, and $\frac{\partial P(T)}{\partial \gamma} = 0$ for some γ_0 such that $P(T) > \frac{1}{n+1}$ for $\gamma = \gamma_0$. Obtaining a simple necessary condition on P(TRS) and P(SRS) such that these restrictions hold seems impossible.

8. Conclusions

a. When small numbers of targets and sweepers are in operation and their distributions are known, numerical analyses such as those in Sections 3 thru 6 are feasible and quite simple. When large numbers of elements must be considered, these methods become cumbersome and eventually impossible to calculate without mechanical aids.

b. Minehunting does not effect the optimum sophistication of a mine but does effect the overall P(T), reducing P(T) as minehunting effectiveness is increased.

c. An increase in sweeper effort with fixed target activity reduces the optimum gate width.

d. A sufficient condition that P(T) have a maximum and thus an optimum gate width is that

$$\lim_{\gamma \to \infty} \frac{\dfrac{\partial P(TRS)}{\partial \gamma}}{\dfrac{\partial P(SRS)}{\partial \gamma}} < 1 .$$

e. There were no necessary conditions of practical significance found that P(T) have a maximum and thus an optimum gate width except that $f_T(\gamma)$ and $f_S(\gamma)$ be continuous, and $\frac{\partial P(T)}{\partial \gamma} = 0$ for some γ_0 such that $P(T) > \frac{1}{n+1}$ for $\gamma = \gamma_0$.

f. The evaluation of the optimum gate width by the methods of Section 7 seems improbable.

g. An interesting extension of this study would be the analysis of actual distributions of targets and sweepers in a given port. From such a study a mine designer can be guided as to the optimal sophistication needed in a mine destined for the given port.